Exercise 1. Suppose you are asked to find minimum and maximum values of a continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ on one of the following regions:

(a) all of $\mathbb{R}^n$,

(b) an open region in $\mathbb{R}^n$ (one example is region $x_1^2 + \ldots + x_n^2 < 1$),

(c) a hypersurface in $\mathbb{R}^n$ comprised of those points satisfying a single constraint equation $g(x_1, \ldots, x_n) = c$ (examples include the hypersurface defined by $x_1^2 + \ldots + x_n^2 = 1$ or $x_1 + \ldots + x_n = 0$), or

(d) a closed and bounded region in $\mathbb{R}^n$ (an example is the region defined by $x_1^2 + \ldots + x_n^2 \leq 1$).

For each of the four types of regions described above: (1) state and defend whether or not a global minimum or global maximum value necessarily exists (either by quoting a theorem from the text that global extrema exist or by giving a specific example of a function that has no global extremum on a particular region type), and (2) explain, in your own words, the procedure for finding the global extrema in the case(s) where they do exist. (definitions, extrema, regions, procedures)

Theorem 7 Global Existence Theorem for Maxima and Minima Let D be closed and bounded in $\mathbb{R}^n$ and let $f\colon D \to \mathbb{R}$ be continuous. Then f assumes its absolute maximum and minimum values at some points $\mathbf{x}_0$ and $\mathbf{x}_1$ of D.

(textbook page 180)

By the above theorem, we know that any continuous function defined on a closed and bounded region D achieves its global max and global min at some points in D.

For type (a), (b), or (c) regions, we can find examples of functions that have no global max or no global min.

Ex for (a)

Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = x$.
The image of this function is all of $\mathbb{R}$. Especially, the function is not bounded above so there is no global max.

Ex 1 for (b)

Let D denote the open unit disc $x^2+y^2<1$ and consider $f(x,y) = y^2 - x^2$.
f is differentiable everywhere on D and has a unique critical point at $(0,0)$.
Using the second derivative test, we can show that the origin is in fact a saddle point. Hence, there is no local max in D and therefore, no global max in D.

Ex 2 for (b)

Again, let D denote the open unit disc $x^2+y^2<1$ and $f: D \longrightarrow \mathbb{R}$ be given by $f(x,y) = x$. The image of f is the open interval $(-1,1)$ in $\mathbb{R}$. We see that f has no global max.

Ex for (c)

Define a hypersurface in $\mathbb{R}^2$ by the equation $x-y=0$ and consider the function $f(x,y) = x$. As a set, the hypersurface $= \{(x,x) \mid x \in \mathbb{R}\}$ and $f(x,x) = x$. We see that the image of the function is all of $\mathbb{R}^2$.
The function is not bounded above, and therefore there is no global max.

Now, let us discuss the procedure for finding global extrema in the cases where they do exist. For closed and bounded regions, the strategy is described in the textbook (page 181), so we'll skip here.

Strategy for (a) and (b).

If f achieves a global extrema on an open set, then that point must be a critical point for f. That is, at that point

- either f is not differentiable
- or f is differentiable and $\nabla f = \vec{0}$.

We compute the values of f at all critical points and find the largest and the smallest.

Strategy for (c).

Let R denote the hypersurface we're considering. If the restricted function $f|_R$ assumes a global extrema at $x_0 \in R$, then x_0 must be a critical point for $f|_R$.

If both f and g are of C^1, then we may collect the points $x_0 \in R$ s.t.

- either $\nabla g(x_0) = \vec{0}$
- or $\nabla g(x_0) \neq \vec{0}$ and $\nabla f(x_0) = \lambda \nabla g(x_0)$ for some $\lambda \in \mathbb{R}$.

We compute the values of f at all critical points and find the largest and the smallest.